# An Introduction to Curve Sketching

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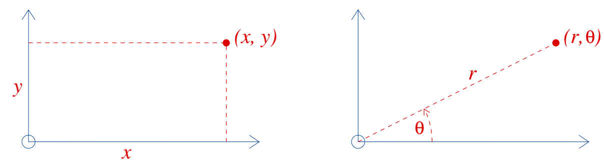
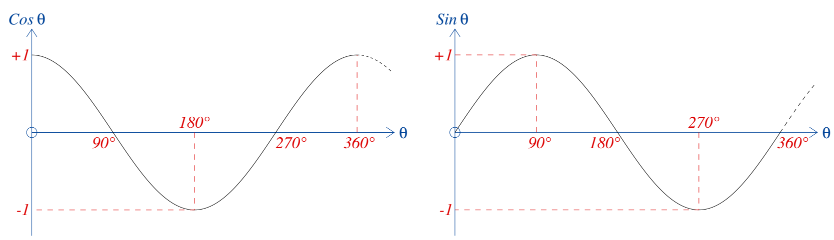
### Cartesian and polar coordinates: brief notes

Recall that a point $P : (x, y)$ in the plane can be represented by giving its horizontal distance $x$ and vertical distance $y$ relative to a fixed origin $O : (0, 0)$. Equivalently a point $P : (r, \theta)$ can be represented by specifying its distance $r$ from the origin together with the angle made by the line $OP$ with the horizontal axis. The coordinates $(x, y)$ are called *Cartesian*, while the coordinates $(r, \theta)$ are called *polar*.




**Question 1** *For a given point $P$, what is the relationship between its Cartesian coordinates $(x, y)$ and its polar coordinates $(r, \theta)$?*

When using Cartesian/polar coordinates, the cosine and sine functions, $\cos\theta$, $\sin\theta$, respectively pop up quite frequently. Their graphs are shown below, as a function of the angle $\theta \in [0, 360]$.




## Sketching curves in polar coordinates

Typically a curve is described by writing the vertical coordinate $y$ as a function of the horizontal coordinate $x$, ie, calculate $y$ given $x$. However we can also specify a curve by calculating $r$ as a function of the angle $\theta$. It is sometimes easier to do this, since its Cartesian representation may be messy!

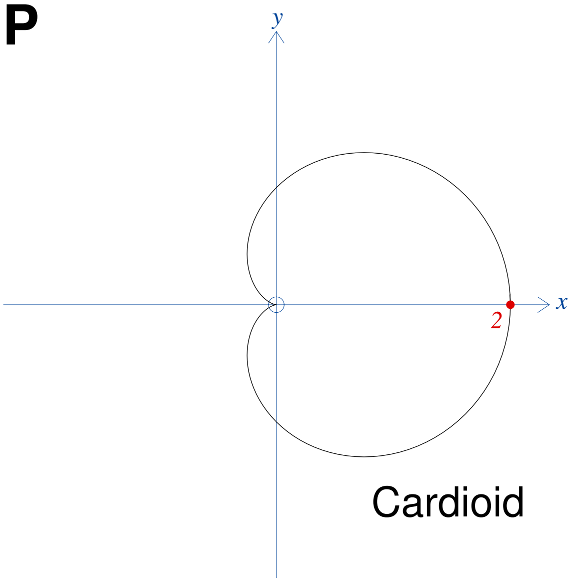
**Question 2** *What is the curve given by the equation $r = 1$? Now, sketch the curve given by the equation $r = 2\cos\theta$.*

**Question 3** *Show that the latter curve, $r = 2\cos\theta$ has the Cartesian representation $(x-1)^2 + y^2 = 1$. What is this curve?*

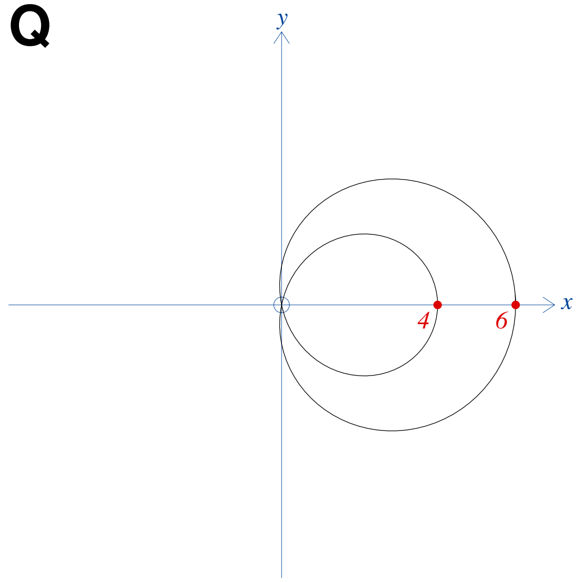
**Exercise 1** *Match the following Polar equations with the 6 curves P,Q,R,S,T,U given on the attached sheet. A) $r = 2$, B) $r = 1 + \frac{1}{2}\cos\theta$, C) $r = 1 + \frac{3}{4}\cos\theta$, D) $r = 1 + \cos\theta$, E) $r = 1 + 5\cos\theta$, F) $r = 1 + 10\cos\theta$.*

**Hint 1** *Usually a calculator is not needed to sketch a curve. You just need to get a* feel *for the shape of the curve, and its distinct features, like its maxima/minima, and where it crosses the axes, etc. The curve may have other features, like* cusps*, we'll say more about this.*

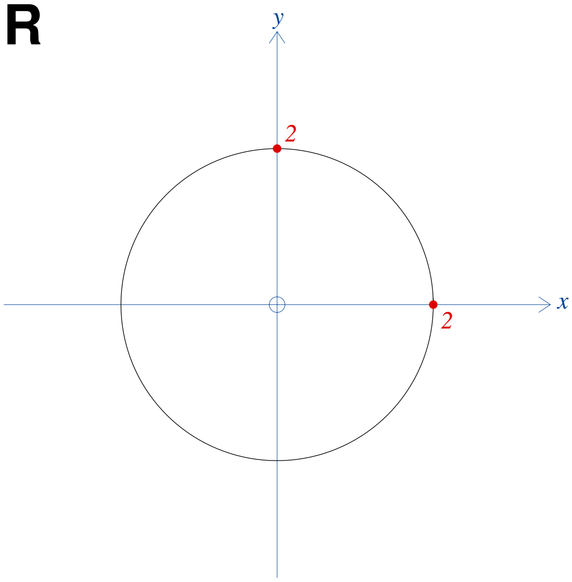
**P**



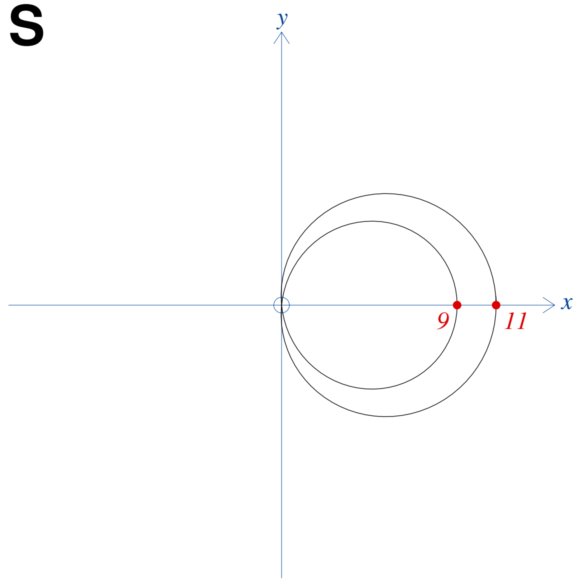

**Q**



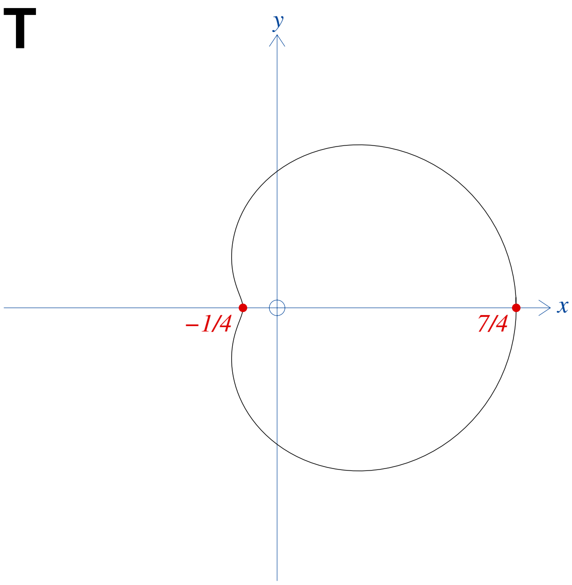

**R**



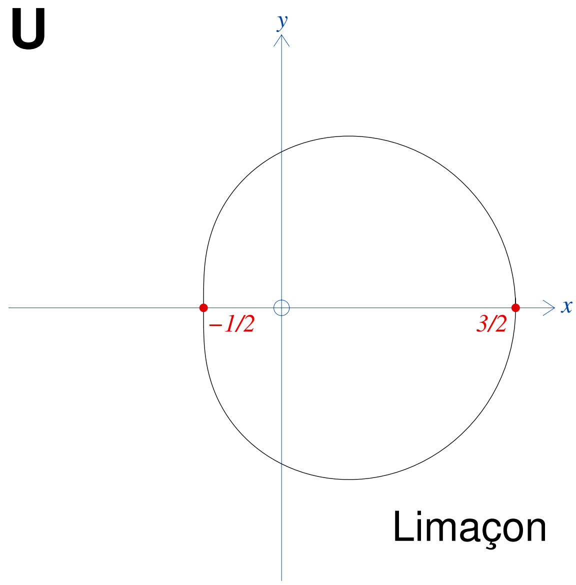

**S**




**T**

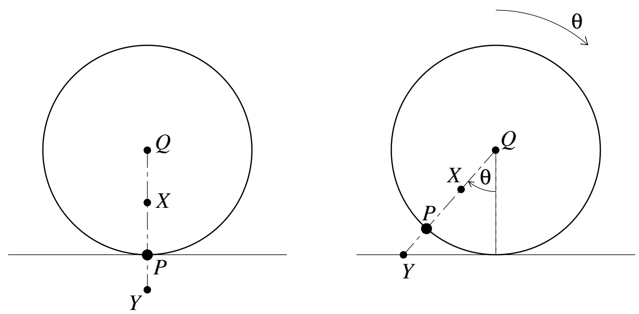



**U**



**Cycloids and trochoids**

Consider a wheel or a disc rolling on a horizontal surface. Let $P$ be a point on the rim, and $Q$ a point at the center. Consider also the straight line joining $P$ to $Q$. Let $X$ be on the line between $P$ and $Q$, and $Y$ a point on the line $PQ$ extended off the disk, see below.




**Question 4** *If $P$ is initially in contact with the surface, and the disc rolls to the right, does $P$ move* up and to the left *or* up and to the right? *Assume the disc does not slip.*

**Exercise 2** *Investigate the paths traced out by $P, Q, X$ and $Y$ as the disc rolls to the right (without slipping). For two revolutions of the disc, sketch the corresponding paths these points make as seen by a still observer.*

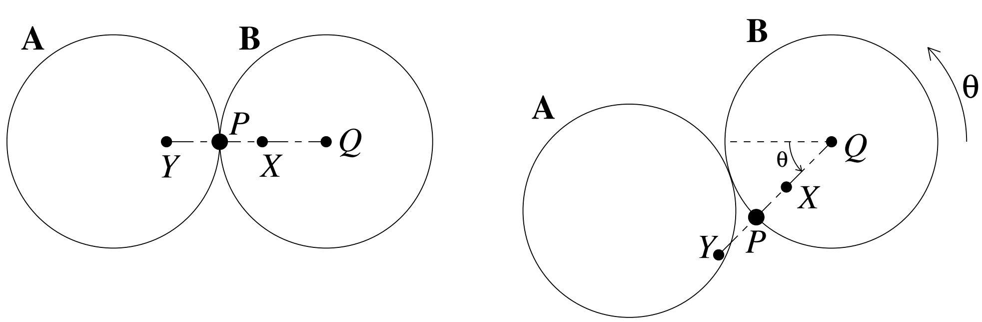
**Hint 2** *The equation of a general point $Z$ on the line $PQ$, (and $PQ$ extended), is given by*

$$x(\theta) = a\left(\frac{2\pi\theta}{360}\right) - b\sin\theta, \quad y(\theta) = a - b\cos\theta,$$

*where $\theta$ is the angle turned through, $x$ and $y$ are the Cartesian coordinates of $Z$, $a$ is the radius of the disc, and $b$ determines how far $Z$ is along the line $PQ$ from $Q$. (eg $b = a$ for $P$, $b < a$ for $X$ and $Q$, and $b > a$ for $Y$.)*

**A disc rolling on a disc: epicycloids**

Consider two disks, one rolling on another. Assume disc A is fixed, while disc B rolls on the rim of A without slipping. Let $P$ be a point on the rim of B, and $Q$ a point at the center of B. Consider also the straight line joining $P$ to $Q$. Let $X$ be on the line between $P$ and $Q$, and $Y$ a point on the line $PQ$ extended off the disk, see below.



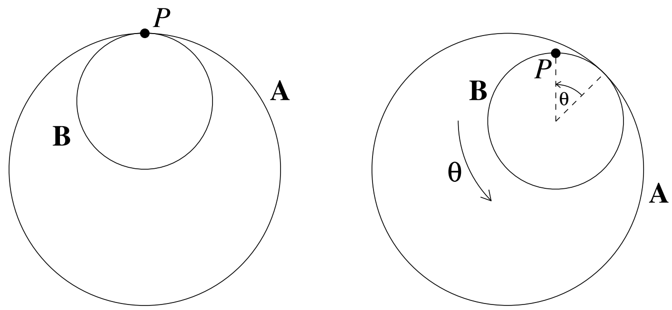

**Exercise 3** *Suppose the radius of A equals the radius of B. Investigate the paths traced out by $P, Q, X$ and $Y$ when disc B rolls around A (without slipping). Sketch the corresponding paths.*

**Exercise 4** *Suppose the radius of disc B = 1/2 radius of disc A. Sketch the paths of $P, Q, X$ and $Y$. What about if the radius of disc B = 1/5 radius of disc A?*

**Exercise 5** *Considering the point $P$ only, what path does $P$ trace out if i) the radius of disc $B = 2/3$ radius of disc A, ii) the radius of disc $B = 2/5$ radius of disc A, iii) the radius of disc $B = 4/7$ radius of disc A. iv) (HARDER), the radius of disc $B = c$ times the radius of disc A, where $c = p/q$ and $p < q$ are integers. v) (CHALLENGE), the radius of disc $B = c$ times the radius of disc A where c is an irrational number (eg $c = 1/\sqrt{5}$).*

### A disc rolling in a disc: hypocycloids

Consider two disks, one rolling inside the other. Assume disc A is fixed, while disc B rolls on the (inside) rim of A without slipping. Just consider the point $P$ on the rim of B, see picture below.




**Exercise 6** *What path does $P$ trace out if i) the radius of disc $B = 1/2$ radius of disc A, ii) the radius of disc $B = 1/4$ radius of disc A, iii) the radius of disc $B = 2/5$ radius of disc A. iv) (HARDER), the radius of disc $B = c$ times the radius of disc A, where $c = p/q$ and $p < q$ are integers. v) (CHALLENGE), the radius of disc $B = c$ times the radius of disc A where c is an irrational number.*

### Rose curves and spirals

**Exercise 7** *Sketch the curves given by the equations (in polar coordinates): i) $r = \sin 2\theta$, ii) $r = \sin 3\theta$, iii), $r = \sin 4\theta$ and iv) $r = \sin 5\theta$ What about for $r = \sin n\theta$, for any integer $n$?*

**Exercise 8** *Sketch the curve given by the equation (in polar coordinates) $r = a\theta$, where $a > 0$ is fixed.*

### Conic sections

Consider the polar equation:

$$r = \frac{1}{1 + e\cos\theta}, \quad e \geq 0.$$

**Exercise 9** *Sketch this polar curve when i) $e = 0$, ii) $0 < e < 1$, iii), $e = 1$ and $e > 1$.*

The curves you obtain are conic sections, namely a circle, an ellipse, a parabola, and a hyperbola.